LETTER TO THE EDITOR

Operational indistinguishabilty of doubly special relativities from special relativity

D V Ahluwalia-Khalilova

Inter-University Centre for Astronomy and Astrophysics (IUCAA), Post Bag 4, Ganeshkhind, Pune 411 007, India‡

E-mail: d.v.ahluwalia-khalilova@heritage.reduaz.mx

Abstract.
We argue that existing doubly special relativities may not be operationally distinguishable from the special relativity. In the process we point out that some of the phenomenologically motivated modifications of dispersion relations, and arrived conclusions, must be reconsidered. Finally, we reflect on the possible conceptual issues that arise in quest for a theory of spacetime with two invariant scales.

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‡ Present address: Department of Mathematics, University of Zacatecas, Zacateacs, ZAC 98060, Mexico

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Introduction.— Postponing discussion on the necessity for relativities with two invariant scales to the concluding section, we first consider recent proposals for such theories. They have been studied under the term doubly special relativity [1, 2]. We show that these suggestions, despite appearances to the contrary, are operationally indistinguishable from the special relativity.

The conclusion is arrived in the following fashion: Since the effect of modified dispersion relations is most dramatically apparent in the kinematical equations of motion we first look at the kinematical description of fermions and bosons in doubly special relativities. At this stage of the argument it would seem that the theory has potentially significant phenomenological implications. But, when we transform the formalism to the recently suggested Judes-Visser variables, we arrive at the conclusion that questions operational distinguishability of these theories from the special relativity.

Doubly Special Relativities of Amelino-Camelia, and Magueijo and Smolin.— Simplest of doubly special relativities (`DSR`)† result from keeping the algebra of boost- and rotation- generators intact while modifying the boost parameter in a non-linear manner. Specifically, in the `DSR` of Amelino-Camelia the boost parameter, $\boldsymbol{\varphi}$, changes from the special relativistic form

$$\cosh\varphi = \frac{E}{m}\,, \quad \sinh\varphi = \frac{p}{m}\,, \quad \widehat{\boldsymbol{\varphi}} = \frac{\mathbf{p}}{p}\,, \tag{1}$$

to [1, 3, 4]

$$\cosh\xi = \frac{1}{\mu}\left(\frac{e^{\ell_P E} - \cosh\left(\ell_P\, m\right)}{\ell_P \cosh\left(\ell_P\, m/2\right)}\right)\,, \tag{2}$$

$$\sinh\xi = \frac{1}{\mu}\left(\frac{p\, e^{\ell_P E}}{\cosh\left(\ell_P\, m/2\right)}\right)\,, \quad \widehat{\boldsymbol{\xi}} = \frac{\mathbf{p}}{p}\,, \tag{3}$$

while for the `DSR` of Magueijo and Smolin the change takes the form [5, 4]

$$\cosh\xi = \frac{1}{\mu}\left(\frac{E}{1-\ell_P\, E}\right)\,, \tag{4}$$

† The phrase "doubly special relativity" is somewhat confusing. The special of "special relativity" refers to the circumstance that one restricts to a special class of inertial observers which move with relative uniform velocity. The general of "general relativity" lifts this restriction. The "special" of special relativity has nothing to do with one versus two invariant scales. It rather refers to the special class of inertial observers; a circumstance that remains unchanged in special relativity with two invariant scales. The theory of general relativity with two invariant scales would thus not be called "doubly general relativity." However, given widespread usage of the term "doubly special relativity" we shall use it here but with the explicit understanding that by it one means a special relativity with two invariant scales.

$$\sinh\xi \;=\frac{1}{\mu}\left(\frac{p}{1-\ell_P\,E}\right)\,,\quad \widehat{\boldsymbol{\xi}}=\frac{\mathbf{p}}{p}\,. \tag{5}$$

Here, μ is a Casimir invariant of $\texttt{DSR}$ (see Eq. (24) below) and is given by

$$\mu=\begin{cases}\dfrac{2}{\ell_P}\,\sinh\left(\dfrac{\ell_P\,m}{2}\right) & \text{for Ref. [1]'s } \texttt{DSR}\\[2ex] \dfrac{m}{1-\ell_P m} & \text{for Ref. [5]'s } \texttt{DSR}\end{cases} \tag{6}$$

The notation is that of Ref. [4]; with the minor exceptions: λ, μ_0, m_0 there are ℓ_P, μ, m here.

Now, it is an assumption of DSR theories that the non-linear action of $\boldsymbol{\xi}$ is restricted to the momentum space *only*. No fully satisfactory spacetime description in the context of the DSR theories has yet emerged, and we are not sure if such an operationally meaningful description indeed exists. Therefore, our arguments shall be made entirely in the momentum space.

Master equation for spin-1/2: Dirac case.— Since the underlying spacetime symmetry generators remain unchanged much of the formal apparatus of the finite dimensional representation spaces associated with the Lorentz group remains intact. In particular, there still exist (1/2, 0) and (0, 1/2) spinors. But now they transform from the rest frame to an inertial frame in which the particle has momentum, $\mathbf{p}$, as:

$$\phi_{(1/2,\,0)}\,(\mathbf{p})\;=\;\exp\left(+\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\xi}\right)\phi_{(1/2,0)}\,(\mathbf{0})\,, \tag{7}$$

$$\phi_{(0,\,1/2)}\,(\mathbf{p})\;=\;\exp\left(-\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\xi}\right)\phi_{(0,1/2)}\,(\mathbf{0})\,. \tag{8}$$

Since the null momentum vector $\mathbf{0}$ is still isotropic, one may assume that (see p. 44 of Ref. [6] *and* Refs. [7, 8, 9]):

$$\phi_{(0,1/2)}\,(\mathbf{0})=\zeta\,\phi_{(1/2,0)}\,(\mathbf{0})\,, \tag{9}$$

where ζ is an undetermined phase factor. In general, the phase ζ encodes C, P, and T properties. The interplay of Eqs. (7-8) and (9) yields the Master equation for the $(1/2,\,0)\oplus(0,\,1/2)$ spinors,

$$\psi\,(\mathbf{p})=\begin{pmatrix}\phi_{(1/2,\,0)}\,(\mathbf{p})\\ \phi_{(0,\,1/2)}\,(\mathbf{p})\end{pmatrix}\,, \tag{10}$$

to be

$$\begin{pmatrix}-\zeta 1_2 & \exp\left(\boldsymbol{\sigma}\cdot\boldsymbol{\xi}\right)\\ \exp\left(-\boldsymbol{\sigma}\cdot\boldsymbol{\xi}\right) & -\zeta^{-1}1_2\end{pmatrix}\psi\,(\mathbf{p})=0\,, \tag{11}$$

where 1_n stands for $n\times n$ identity matrix (and 0_n represents the corresponding null matrix). This is one of the central results on which would be anchored, as we would see latter, the thesis summarized in the Abstract.

As a check, taking $\boldsymbol{\xi}$ to be $\boldsymbol{\varphi}$, and after some simple algebraic manipulations, the Master equation (11) reduces to:

$$\begin{pmatrix}-m\zeta 1_2 & E1_2+\boldsymbol{\sigma}\cdot\mathbf{p}\\ E1_2-\boldsymbol{\sigma}\cdot\mathbf{p} & -m\zeta^{-1}1_2\end{pmatrix}\psi\,(\mathbf{p})=0\,. \tag{12}$$

With the given identification of the boost parameter we are in the realm of special relativity. There, the operation of parity is well understood. Demanding parity covariance for Eq. (12), we obtain $\zeta = \pm 1$. Identifying

$$\left(\begin{array}{cc} 0_2 & 1_2 \\ 1_2 & 0_2 \end{array}\right), \quad \left(\begin{array}{cc} 0_2 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0_2 \end{array}\right), \tag{13}$$

with the Weyl-representation γ^0, and γ^i, respectively; Eq. (12) reduces to the Dirac equation of special relativity,

$$\left(\gamma^\mu p_\mu \mp m\right) \psi\left(\mathbf{p}\right) = 0\,. \tag{14}$$

The linearity of the Dirac equation in $p_\mu = (E, -\mathbf{p})$, is now clearly seen to be associated with two observations:

$\mathcal{O}_1$. That, $\boldsymbol{\sigma}^2 = 1_2$; and

$\mathcal{O}_2$. That in special relativity, the hyperbolic functions – see Eq. (1) – associated with the boost parameter are linear in p_μ.

In DSR, observation $\mathcal{O}_1$ still holds. But, as Eqs. (2 - 5) show, $\mathcal{O}_2$ is strongly violated. The extension of the presented formalism for Majorana spinors is more subtle [10, 11, 12] and can be carried using the techniques developed in Ref. [13].

Master equation for higher spins.— The above-outlined procedure applies to all, bosonic as well as fermionic, $(j, 0) \oplus (0, j)$ representation spaces. It is not confined to $j = 1/2$. A straightforward generalization of the $j = 1/2$ analysis immediately yields the Master equation for an arbitrary-spin,

$$\left(\begin{array}{cc} -\zeta\, 1_{2j+1} & \exp\left(2\mathbf{J}\cdot\boldsymbol{\xi}\right) \\ \exp\left(-2\mathbf{J}\cdot\boldsymbol{\xi}\right) & -\zeta^{-1}\, 1_{2j+1} \end{array}\right) \psi\left(\mathbf{p}\right) = 0\,, \tag{15}$$

where

$$\psi\left(\mathbf{p}\right) = \left(\begin{array}{c} \phi_{(j,0)}\left(\mathbf{p}\right) \\ \phi_{(0,j)}\left(\mathbf{p}\right) \end{array}\right). \tag{16}$$

Equation (15) contains the central result of the previous section as a special case. For studying the special relativistic limit it is convenient to bifurcate the $(j, 0) \oplus (0, j)$ space into two sectors by splitting the $2(2j + 1)$ phases, ζ, into two sets: $(2j + 1)$ phases ζ_+, and the other $(2j + 1)$ phases ζ_-. Then in particle's rest frame the $\psi(\mathbf{p})$ may be written as:

$$\psi_h(\mathbf{0}) = \begin{cases} u_h(\mathbf{0}) & \text{when } \zeta = \zeta_+ \\ v_h(\mathbf{0}) & \text{when } \zeta = \zeta_- \end{cases} \tag{17}$$

The explicit forms of $u_h(\mathbf{0})$ and $u_h(\mathbf{0})$ (see Eq. (9)) are:

$$u_h(0) = \left(\begin{array}{c} \phi_h(\mathbf{0}) \\ \zeta_+\, \phi_h(\mathbf{0}) \end{array}\right), \; v_h(0) = \left(\begin{array}{c} \phi_h(\mathbf{0}) \\ \zeta_-\, \phi_h(\mathbf{0}) \end{array}\right), \tag{18}$$

where the $\phi_h(\mathbf{0})$ are defined as: $\mathbf{J}\cdot\widehat{\mathbf{p}}\,\phi_h(\mathbf{0}) = h\,\phi_h(\mathbf{0})$, and $h = -j, -j+1, \ldots, +j$. In the parity covariant special relativistic limit, we find $\zeta_+ = +1$ while $\zeta_- = -1$.

As a check, for $j = 1$, identification of $\boldsymbol{\xi}$ with $\boldsymbol{\varphi}$, and after implementing parity covariance, Eq. (15) yields

$$\left(\gamma^{\mu\nu} p_\mu p_\nu \mp m^2\right) \psi(\mathbf{p}) = 0\,. \tag{19}$$

The $\gamma^{\mu\nu}$ are unitarily equivalent to those of Ref. [15], and thus we reproduce *bosonic matter fields* with $\{C,\, P\} = 0$. A carefully taken massless limit then shows that the resulting equation is consistent with the free Maxwell equations of electrodynamics.

Since the $j = 1/2$ and $j = 1$ representation spaces of `DSR` reduce to the Dirac and Maxwell descriptions, it would seem apparent (and as is often argued in similar contexts [16] – wrongly as we will soon see –) that the `DSR` contains physics beyond the linear-group realizations of special relativity. To the lowest order in ℓ_P, Eq. (11) yields

$$\left(\gamma^\mu p_\mu + \tilde{m} + \delta_1\, \ell_P\right) \psi(\mathbf{p}) = 0\,, \tag{20}$$

where

$$\tilde{m} = \begin{pmatrix} -\zeta 1_2 & 0_2 \\ 0_2 & -\zeta^{-1} 1_2 \end{pmatrix} m \tag{21}$$

and

$$\delta_1 = \begin{cases} \gamma^0 \left(\dfrac{E^2 - m^2}{2}\right) + \gamma^i p_i\, E & \text{for Ref. [1]'s } \texttt{DSR} \\ \gamma^\mu p_\mu\, (E - m) & \text{for Ref. [5]'s } \texttt{DSR} \end{cases} \tag{22}$$

Similarly, the presented Master equation can be used to obtain `DSR`'s counterparts for Maxwell's electrodynamic. Unlike the Coleman-Glashow framework [17], the existing DSRs provide *all* corrections, say, to the standard model of the high energy physics, in terms of *one* – and *not forty six* – fundamental constant, ℓ_P. Had DSRs been operationally distinct this would have been a a remarkable power of DSR-motivated frameworks.

Challenging the DSR's operational distinguishability from Special Relativistic framework: Spin-1/2 and Spin-1 description in Judes-Visser Variables.— We now show that the DSR program as implemented currently is misleading. The question is what are the operationally measurable quantities in DSR? The E is no longer the 0th component, nor is $\mathbf{p}$ the spatial component of 4−momentum. Neither is m an invariant under the DSR boosts. Their physical counterparts, as we interpret them, are Judes-Visser variables [4], $\eta^\mu \equiv (\epsilon(E,p),\, \boldsymbol{\pi}(E,p)) = (\eta^0, \boldsymbol{\eta})$, and μ. The $\epsilon(E,p)$ and $\boldsymbol{\pi}(E,p)$ relate to the rapidity parameter $\boldsymbol{\xi}$ of `DSR` in same functional form as do E and $\mathbf{p}$ to $\boldsymbol{\varphi}$ of special relativity:

$$\cosh(\xi) = \frac{\epsilon(E,p)}{\mu}\,, \quad \sinh(\xi) = \frac{\pi(E,p)}{\mu}\,, \tag{23}$$

where

$$\mu^2 = \left[\epsilon(E,p)\right]^2 - \left[\boldsymbol{\pi}(E,p)\right]^2\,. \tag{24}$$

They provide the most economical and physically transparent formalism for representation space theory in DSR. For $j = 1/2$ and $j = 1$, Eq. (15) yields the *exact DSR* equations for $\psi(\boldsymbol{\pi})$:

$$\left(\gamma^\mu \eta_\mu + \tilde{\mu}\right)\psi\left(\boldsymbol{\pi}\right) = 0\,, \quad \text{where}\ \ \tilde{\mu} = \begin{pmatrix} -\zeta^{-1} 1_2 & 0_2 \\ 0_2 & -\zeta 1_2 \end{pmatrix} \mu\,, \tag{25}$$

$$\left(\gamma^{\mu\nu} \eta_\mu \eta_\nu + \tilde{\mu}^2\right)\psi(\boldsymbol{\pi}) = 0\,, \quad \text{with}\ \ \tilde{\mu}^2 = \begin{pmatrix} -\zeta^{-1} 1_3 & 0_3 \\ 0_3 & -\zeta 1_3 \end{pmatrix} \mu^2\,. \tag{26}$$

From an operational point of view the η^μ and μ *are* the physical observables. The old operational meaning of the symbols E and $\mathbf{p}$ is lost in the non-linear realization of the boost in the momentum space. There is covariance of the form of the considered‖ fermionic and bosonic wave equations under the transformations:

$$m \to \mu\,, \quad p^\mu \to \eta^\mu\,, \quad \boldsymbol{\varphi} \to \boldsymbol{\xi}\,. \tag{27}$$

Thus, DSR and special relativistic descriptions of fermions and bosons become operationally indistinguishable.

Towards a new relativity.—

The program of DSRs as presently formulated in terms of “phenomenologically motivated” dispersion relations seems untenable to us. Our reasons are as follows. In going from the Galilean to special relativity it is true that one needed a new scale. It was, as we know, the speed of light, c. It unified space and time into one entity. Now if one wishes to introduce another invariant scale, such as ℓ_p, then the question arises as to what aspect of spacetime it purports to unify, or is there a new interaction which it brings in the realm of spacetime structure. Recall that a new aspect in spacetime did emerge with the theory of general relativity which endowed spacetime with the dynamics of the gravitational field. With the work of Bekenstein and Hawking [18, 19, 20], one fused quantum and thermodynamical features into the framework. Nothing similar seems to have been addressed in DSRs, and unless that happens we are afraid these suggestions would remain devoid of any physical content.

Since the possible existence of ultra high energy cosmic rays beyond the GZK cutoff [21, 22] already point to a possible violation of the Lorentz symmetry and emergence of new dispersion relation[s] [23, 24, 25, 26, 27, 29], it should be asked if there exists a natural mechanism which asks for a new relativity. One such possibility has unexpectedly emerged in a recent work on Majorana spinors [13]. The essential idea, for spin one half, is to promote the phase ζ to a complex 2×2 phase matrix with determinant ±1. Its precise form is then fixed by demanding that the resulting spinors have some pre assigned property under one of the discrete operators: C, P, and T. Preliminary analysis of Ref. [13] suggests that in this process one obtains new dispersion relations which demand specific modifications in our conception of spacetime. Since such a global relative phase – or, phase matrix – encodes C, P, and T properties of matter fields

‖ The result is expected to be the same for other representation spaces.

without directly becoming source of energy momentum tensor, the spacetime remains flat. But, now it incorporates the new quantum phase aspect into its structure and leads to the identification of a preferred frame in the context of Majorana particles [13, 14]. In contrast to DSRs, in the considered framework there is a clear picture of what is purported to be synthesized – the extension to special relativity is now sought in the same spirit as was done to get a theory of gravitation in general relativity. The new aspect emerges due to introduction of a C, P, and T encoding phase field. It appears, at present, that in such a theory the Dirac particles are associated with the standard dispersion relation while a new dispersion relation emerges for the Majorana-like sector and takes the form: $E = 2m \pm \sqrt{\mathbf{p}^2 + m^2}$. Eventually, in presence of gravitation, the new phase field may have to be made local.

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